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Saturation of Trigonometric Polynomial Operators

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1. INTRODUCTION

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic continuous functions on the real line with the supremum norm $\|\cdot\|$. Let $\{L_n\}$ be a sequence of bounded linear operators of $C_{2\pi}$ into itself. Suppose that there exists a sequence $\{\phi_n\}$ of positive numbers converging to zero such that every f in $C_{2\pi}$ for which $\| L_n(f) - f\| = o(\phi_n)$ is a constant function, and there exists a nonconstant function f_0 in $C_{2\pi}$ such that $\| L_n(f_0) - f_0 \| = \mathcal{O}(\phi_n)$. Then the sequence $\{L_n\}$ is said to be saturated with the order $\{\phi_n\}$ and the class $\mathscr{S}(L_n)$, consisting of all f in $C_{2\pi}$ for which $\| L_n(f) - f\| = \mathcal{O}(\phi_n)$, is called the saturation class.

The problems of saturation have been investigated by several authors; an excellent source for references and a systematic treatment of theorems of saturation can be found in Butzer and Nessel [2] and DeVore [5]. Saturation theory in an arbitrary Banach space setting is treated by Butzer, Nessel, and Trebels [3, 4].

Here we are concerned with trigonometric polynomial operators which can be defined as follows. Let $(\lambda(n; k))_{n,k\geqslant 1}$ be a lower triangular matrix, that is, an infinite real matrix satisfying $\lambda(n; k) = 0$ whenever $k > n$. Let $f \in C_{2\pi}$, let its Fourier series be

$$S[f] = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x) \tag{1}$$

and define

$$T_n(f)(x) = \sum_{k=0}^{n} \lambda(n; k)\, A_k(x), \tag{2}$$

where $\lambda(n; 0) = 1$.

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Let $\tilde{s}_n(f)(x)$ denote the nth partial sum of the conjugate series of (1)

$$\tilde{S}[f] = \sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx).$$

Then we have

$$\tilde{s}_n(f)(x) = (1/\pi) \int_0^{\pi} \{f(x+t) - f(x-t)\} \tilde{D}_n(t)\, dt,$$

where

$$\tilde{D}_n(t) = \{\cos(t/2) - \cos(2n+1)(t/2)\}/2 \sin(t/2).$$

We say that

$$\tilde{f}(x) = (1/2\pi) \int_0^{\pi} \{f(x+t) - f(x-t)\} \cot(t/2)\, dt \tag{3}$$

is the conjugate function of f, if the integral on the right-hand side of (3) converges absolutely for all x and if

$$\int_0^{\pi} |f(x+t) - f(x-t)| \cot(t/2)\, dt$$

is an integrable function.

The purpose of this paper is to establish a saturation theorem for the sequence $\{T_n\}$ of operators on $C_{2\pi}$ defined by (2); applications are made to Nörlund (= Voronoi) means whose saturation problem is dealt with by Buchwalter [1], Goel, *et al.* [6], Tureckii [7, 8], and Zuk [9].

2. A Saturation Theorem

We have the following saturation theorem:

THEOREM 1. *Suppose that there exists a sequence* $\{\phi_n\}$ *of positive real numbers converging to zero, which satisfies*

$$\lim_{n\to\infty} (1 - \lambda(n; k))/\phi_n = k \qquad (k = 1, 2, 3,...) \tag{4}$$

and

$$\sum_{k=0}^{n} |\Lambda(n; k)| = \mathcal{O}(\phi_n), \tag{5}$$

where $\Lambda(n; k) = \lambda(n; k) - 2\lambda(n; k+1) + \lambda(n; k+2)$. *Then* $\{T_n\}$ *is saturated with the order* $\{\phi_n\}$ *and* $\mathscr{S}(T_n) = \{f \in C_{2\pi}; \tilde{f} \in \mathrm{Lip}\ 1\}$.

Proof. The proof requires only to show that under assumption (5), $\tilde{f} \in \mathrm{Lip}\ 1$ implies $\| T_n(f) - f \| = \mathcal{O}(\phi_n)$.

Set

$$G_n(t) = \sum_{k=0}^{n} \Lambda(n; k) \int_t^{\pi} (\sin(k+1)\, u)/u^2\, du \qquad (0 \leqslant t \leqslant \pi).$$

Then we have

$$\int_0^{\pi} | G_n(t)|\, dt = \mathcal{O}\left[\sum_{k=0}^{n} | \Lambda(n; k)|\right]. \tag{6}$$

Indeed, since

$$\begin{aligned}\int_{(k+1)t}^{(k+1)\pi} (\sin x)/x^2\, dx &= \mathcal{O}(\log(1/(k+1)\, t)) \qquad ((k+1)\, t < 1)\\ &= \mathcal{O}(1/(k+1)^2\, t^2) \qquad ((k+1)\, t \geqslant 1),\end{aligned}$$

we have

$$\begin{aligned}G_n(t) &= \sum_{k=0}^{n} \Lambda(n; k)(k+1) \int_{(k+1)t}^{(k+1)\pi} (\sin x)/x^2\, dx\\ &= \sum_{(k+1)t<1} \Lambda(n; k)(k+1) \int_{(k+1)t}^{(k+1)\pi} (\sin x)/x^2\, dx\\ &\quad + \sum_{(k+1)t\geqslant 1} \Lambda(n; k)(k+1) \int_{(k+1))}^{(k+1)\pi} (\sin x)/x^2\, dx,\end{aligned}$$

and so

$$\begin{aligned}| G_n(t)| = \mathcal{O}\Big[&\sum_{(k+1)t<1} | \Lambda(n; k)|\, (k+1) \log(1/(k+1)\, t)\\ &+ \sum_{(k+1)t\geqslant 1} | \Lambda(n; k)|\, (k+1)/(k+1)^2\, t^2\Big].\end{aligned}$$

Thus we have

$$\begin{aligned}\int_0^{\pi} | G_n(t)|\, dt = \mathcal{O}\Big[\int_0^{\pi} \Big\{&\sum_{(k+1)t<1} | \Lambda(n; k)|\, (k+1) \log(1/(k+1)\, t)\\ &+ \sum_{(k+1)t\geqslant 1} | \Lambda(n; k)|/(k+1)\, t^2\Big\}\, dt\Big]\end{aligned}$$

$$= \mathcal{O}\left[\sum_{k=0}^{n} | \Lambda(n; k)| \left\{\int_0^{1/(k+1)} (k+1) \log(1/(k+1)\, t)\, dt \right.\right.$$

$$\left.\left. + \int_{1/(k+1)}^{\pi} 1/(k+1)\, t^2\, dt\right\}\right]$$

$$= \mathcal{O}\left[\sum_{k=0}^{n} | \Lambda(n; k)| \{1 + (k + 1 - 1/\pi)/(k+1)\}\right]$$

$$= \mathcal{O}\left[\sum_{k=0}^{n} | \Lambda(n; k)|\right].$$

Now we have

$$T_n(\tilde{s}_n(\tilde{f}))(x) = -\sum_{k=1}^{n} \lambda(n; k)\, A_k(x)$$

$$= \sum_{k=0}^{n} \{\lambda(n; k) - \lambda(n; k+1)\}\, \tilde{s}_k(\tilde{f})(x)$$

$$= \sum_{k=0}^{n} \{\lambda(n; k) - \lambda(n; k+1)\}(1/2\pi)$$

$$\times \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \cot(t/2)\, dt$$

$$- \sum_{k=0}^{n} \{\lambda(n; k) - \lambda(n; k+1)\}(1/2\pi)$$

$$\times \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \cos(2k+1)(t/2)/\sin(t/2)\, dt$$

$$= (-f(x) + a_0/2) - \sum_{k=0}^{n} \{\lambda(n; k) - \lambda(n; k+1)\}(1/2\pi)$$

$$\times \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \cos(2k+1)(t/2)/\sin(t/2)\, dt.$$

Thus we have

$$T_n(f)(x) - f(x) = (1/2\pi) \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\, \Psi_n(t)\, dt,$$

where

$$\Psi_n(t) = (1/\sin(t/2)) \sum_{k=0}^{n} \{\lambda(n; k) - \lambda(n; k+1)\} \cos(2k+1)(t/2).$$

Since

$$\Psi_n(t) = (1/2\sin^2(t/2)) \sum_{k=0}^{n} \Lambda(n; k)\sin(k+1)t$$

$$= (2/t^2 + \mathcal{O}(1)) \sum_{k=0}^{n} \Lambda(n; k)\sin(k+1)t$$

$$= (2/t^2) \sum_{k=0}^{n} \Lambda(n; k)\sin(k+1)t + \mathcal{O}(\phi_n)$$

and $\tilde{f} \in \mathrm{Lip}\ 1$, we obtain

$$T_n(f)(x) - f(x) = (1/\pi) \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\}$$

$$\times \sum_{k=0}^{n} \Lambda(n; k) \frac{\sin(k+1)t}{t^2}\, dt + \mathcal{O}(\phi_n)$$

$$= -(1/\pi) \int_0^{\pi} \{g(x+t) + g(x-t)\}\, G_n(t)\, dt + \mathcal{O}(\phi_n),$$

where g is a bounded integrable function with period 2π.

Therefore, by (6), we have

$$|T_n(f)(x) - f(x)| = \mathcal{O}\left[\int_0^{\pi} |G_n(t)|\, dt\right] + \mathcal{O}(\phi_n)$$

$$= \mathcal{O}(\phi_n) + \mathcal{O}(\phi_n) = \mathcal{O}(\phi_n),$$

which yields

$$\| T_n(f) - f \| = \mathcal{O}(\phi_n).$$

Because of assumption (4) the rest of the conclusion of the theorem follows from [2, Theorems 12.1.3 and 12.1.4]. Thus the theorem is proved.

COROLLARY. *If* (4) *and* (5) *in Theorem* 1 *are satisfied for* $\phi_n = |1 - \lambda(n; 1)|$, *then* $\{T_n\}$ *is saturated with the order* $\{|1 - \lambda(n; 1)|\}$ *and* $\mathcal{S}(T_n) = \{f \in C_{2\pi} ; \tilde{f} \in \mathrm{Lip}\ 1\}$.

Remark. Condition (5) should be compared with the condition

$$\sum_{k=0}^{n} (k+1) \left| \Delta^2 \frac{1 - \lambda(n; k)}{k} \right| = \mathcal{O}(\phi_n),$$

where

$$\Delta^2 \frac{1-\lambda(n;k)}{k}$$
$$= \frac{1-\lambda(n;k)}{k} - \frac{2(1-\lambda(n;k+1))}{k+1} + \frac{1-\lambda(n;k+2)}{k+2}$$
$$= -\frac{1}{k+1}\left\{\Delta(n;k) + \frac{1-\lambda(n;k+2)}{k+2} - \frac{1-\lambda(n;k)}{k}\right\},$$

which is the usual condition in saturation theory based on quasi-convexity (cf. [2, (12.2.4) and Corollary 6.3.9, 4]).

3. Application to Nörlund Operators

Let $\{p_n\}_{n\geqslant 1}$ be a sequence of real numbers such that

$$P_n = p_1 + p_2 + p_3 + \cdots + p_n \neq 0.$$

The Nörlund operator N_n in $C_{2\pi}$ is defined by

$$N_n(f)(x) = (1/P_n) \sum_{k=0}^{n} P_{n-k} A_k(x),$$

where $P_0 = 0$, which can be obtained by taking the numbers $\lambda(n;k) = P_{n-k}/P_n$ in (2). Note that if

$$\lim_{n\to\infty} P_{n-k}/P_n = 1 \qquad (k = 1, 2, 3,\ldots),$$

and if

$$\sum_{k=0}^{n-1} (k+1) \mid p_{n-k} - p_{n-k-1} \mid = \mathcal{O}(\mid P_n \mid),$$

where $p_0 = 0$, then $\lim_{n\to\infty} \| N_n(f) - f \| = 0$ for every f in $C_{2\pi}$.

As an immediate consequence of the corollary, we have the following saturation theorem of the sequence $\{N_n\}$ of Nörlund operators, which should be compared with the result in [6].

THEOREM 2. *Suppose that* $p_n \neq 0$ $(n = 1, 2, 3,\ldots)$, $\lim_{n\to\infty} p_n/P_n = 0$,

$$\lim_{n\to\infty} \operatorname{sign}(P_n) \sum_{i=0}^{k-1} p_{n-i}/\mid p_n \mid = k \qquad (k = 1, 2, 3,\ldots) \tag{7}$$

and

$$\sum_{k=1}^{n} | p_k - p_{k-1} | = \mathcal{O}(| p_n |). \tag{8}$$

Then $\{N_n\}$ *is saturated with the order* $\{| p_n/P_n |\}$ *and* $\mathscr{S}(N_n) = \{f \in C_{2\pi}; \tilde{f} \in \mathrm{Lip}\ 1\}$.

Proof. Take $\lambda(n; k) = P_{n-k}/P_n$. Then it can be seen that

$$1 - \lambda(n; k) = (1/P_n) \sum_{i=0}^{k-1} p_{n-i}$$

and

$$\Lambda(n; k) = (p_{n-k} - p_{n-k-1})/P_n .$$

Thus we have

$$(1 - \lambda(n; k))/| 1 - \lambda(n; 1)| = \mathrm{sign}(P_n) \sum_{i=0}^{k-1} p_{n-i}/| p_n |,$$

and so the desired result follows from the Corollary.

Remark. Observe that (7) is equivalent to

$$\lim_{n\to\infty} \mathrm{sign}(P_n)\, p_{n-i}/| p_n | = 1 \qquad (i = 0, 1, 2,...)$$

and that, if $p_{n+1} \geqslant p_n > 0$ $(n = 1, 2, 3,...)$, then (8) is automatically satisfied.

Finally, we mention some examples of $\lambda(n; k)$ $(= P_{n-k}/P_n)$ and ϕ_n $(= | p_n/P_n |)$, respectively.

(i) $\lambda(n; k) = (n - k)/n, \qquad \phi_n = 1/n$

for $p_n = 1$ (in this case, the operator N_n coincides with the nth Cesaro mean operator (of order 1)).

(ii) $\lambda(n; k) = \dfrac{(n - k)(n - k + 1)}{n(n + 1)}, \qquad \phi_n = \dfrac{2}{n + 1}$

for $p_n = n$.

(iii) $\lambda(n; k) = \dfrac{(n - k)(n - k + 1)(n - k + 2)}{n(n + 1)(n + 2)}, \qquad \phi_n = \dfrac{3}{n + 2}$

for $p_n = n(n + 1)$.

(iv) $\lambda(n; k) = \dfrac{(n - k)(n - k + 1)\{2(n - k) + 1\}}{n(n + 1)(2n + 1)},$

$$\phi_n = \frac{6n}{(n + 1)(2n + 1)}$$

for $p_n = n^2$.

$$\text{(v)}\quad \lambda(n; k) = \left\{\frac{(n-k)(n-k+1)}{n(n+1)}\right\}^2, \qquad \phi_n = \frac{4n}{(n+1)^2}$$

for $p_n = n^3$.

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REFERENCES

1. H. BUCHWALTER, Saturation dans un espace normé, *C. R. Acad. Sci. Paris* **250** (1960), 651–653.
2. P. L. BUTZER AND R. J. NESSEL, "Fourier Analysis and Approximation," Vol. I, Birkhäuser, Basel, and Academic Press, New York, 1971.
3. P. L. BUTZER, R. J. NESSEL AND W. TREBELS, On summation processes of Fourier expansions in Banach spaces. I. Comparison theorems, *Tôhoku Math. J.* **24** (1972), 127–140.
4. P. L. BUTZER, R. J. NESSEL AND W. TREBELS, On summation processes of Fourier expansions in Banach spaces. II. Saturation theorems, *Tôhoku Math. J.* **24** (1972), 551–569.
5. R. A. DEVORE, "The Approximation of Continuous Functions by Positive Linear Operators," Lecture Notes in Mathematics No. 293, Springer-Verlag, Berlin/Heidelberg/New York, 1971.
6. D. S. GOEL, A. S. B. HOLLAND, C. NASIM, AND B. N. SAHNEY, Best approximation by a saturation class of polynomial operators, *Pacific J. Math.* **55** (1974), 149–155.
7. A. H. TURECKII, The class of saturation for the "Logarithmic" mean method of summing Fourier series (Russian), *Dokl. Akad. Nauk BSSR* **4** (1960), 95–100.
8. A. H. TURECKII, Saturation classes in a space *C* (Russ.), *Izv. Akad. Nauk SSSR Ser. Mat.* **25** (1961), 411–442.
9. V. V. ZUK, The approximation of a 2π-periodic function by a linear method (Russ.), *Leningrad. Meh. Inst. Sb. Naučn. Trudov* **50** (1965), 93–115.